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6. AUTHORS Robert M. Kirby, Robert Haimes			5d. PROJECT NUMBER		
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14. ABSTRACT The discontinuous Galerkin method (DGM) has become, in recent times, one of the most widely researched and utilized discretization methodologies for solving problems in science and engineering. Fundamentally based upon the mathematical framework of variational methods, the DG methodology provides hope that computationally fast, efficient and robust methods can be constructed for solving real-world problems. Through a combination of a dual path to convergence, allowing naturally both conforming and non-conforming (hanging node) non-overlapping					
15. SUBJECT TERMS Discontinuous Galerkin Method (DGM), high-order methods, visualization, simulation, computation					
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a. REPORT UU	b. ABSTRACT UU	c. THIS PAGE UU			Robert Kirby
					19b. TELEPHONE NUMBER 801-/58-5342

Report Title

Final Report: Visualization of Discontinuous Galerkin Based High-Order Methods

ABSTRACT

The discontinuous Galerkin method (DGM) has become, in recent times, one of the most widely researched and utilized discretization methodologies for solving problems in science and engineering. Fundamentally based upon the mathematical framework of variational methods, the DG methodology provides hope that computationally fast, efficient and robust methods can be constructed for solving real-world problems. Through a combination of a dual path to convergence – allowing naturally both conforming and non-conforming (hanging node) non-overlapping discretizations of the solution domain combined with (possibly non-uniform) polynomial enrichment (also known as p-refinement) – the DG methodology provides a rich mathematical starting point for the development of domain-specific solvers. By not requiring that the solution be continuous across element boundaries, the DGM provides a flexibility that can be exploited both for geometric and solution adaptivity and for parallelization.

The goal of this proposal is to research, develop and implement visualization techniques that are specifically designed to respect the mathematical nature of high-order DGM simulation fields. As our intended use for this research is both simulation debugging and scientific exploratory visualization, we seek methods and implementations that act on high-order DG data directly, that are interactive, and that have quantifiable visual error. This effort is designed to build upon our previous research into the visualization of high-order (continuous Galerkin) finite element fields, but presents a distinct challenge in that we must now consider the additional computational (e.g. precision) and visualization challenges of element-wise discontinuities.

Enter List of papers submitted or published that acknowledge ARO support from the start of the project to the date of this printing. List the papers, including journal references, in the following categories:

(a) Papers published in peer-reviewed journals (N/A for none)

<u>Received</u>	<u>Paper</u>
08/19/2015 2.00	Tiago Etienne, Daniel Jonsson, Timo Ropinski, Carlos Scheidegger, Joao L. D. Comba, Luis Gustavo Nonato, Robert M. Kirby, Anders Ynnerman, Claudio T. Silva. Verifying Volume Rendering Using Discretization Error Analysis, IEEE Transactions on Visualization and Computer Graphics, (01 2014): 0. doi: 10.1109/TVCG.2013.90
08/19/2015 3.00	Blake Nelson, Robert M. Kirby, Robert Haimes. GPU-Based Volume Visualization from High-Order Finite Element Fields, IEEE Transactions on Visualization and Computer Graphics, (01 2014): 70. doi: 10.1109/TVCG.2013.96
08/19/2015 4.00	Blake Nelson, Robert M. Kirby, Steven Parker. Algorithm 940: Optimal Accumulator-Based Expression Evaluation through the Use of Expression Templates, ACM Transactions on Mathematical Software, (04 2014): 21. doi: 10.1145/2591005
TOTAL:	3

Number of Papers published in peer-reviewed journals:

(b) Papers published in non-peer-reviewed journals (N/A for none)

Received Paper

TOTAL:

Number of Papers published in non peer-reviewed journals:

(c) Presentations

Number of Presentations: 0.00

Non Peer-Reviewed Conference Proceeding publications (other than abstracts):

Received Paper

TOTAL:

Number of Non Peer-Reviewed Conference Proceeding publications (other than abstracts):

Peer-Reviewed Conference Proceeding publications (other than abstracts):

Received Paper

07/15/2014 1.00 Shankar P. Sastry , Robert M. Kirby. On Interpolation Errors over Quadratic Nodal Triangular Finite Elements,
International Meshing Roundtable (IMR). 01-OCT-13, . : ,

TOTAL: 1

Number of Peer-Reviewed Conference Proceeding publications (other than abstracts):

(d) Manuscripts

Received Paper

TOTAL:

Number of Manuscripts:

Books

Received Book

TOTAL:

Received Book Chapter

TOTAL:

Patents Submitted

Patents Awarded

Awards

Graduate Students

<u>NAME</u>	<u>PERCENT SUPPORTED</u>	Discipline
Harshithaparnandi Venkata	0.57	
FTE Equivalent:	0.57	
Total Number:	1	

Names of Post Doctorates

<u>NAME</u>	<u>PERCENT SUPPORTED</u>
FTE Equivalent:	
Total Number:	

Names of Faculty Supported

<u>NAME</u>	<u>PERCENT SUPPORTED</u>	National Academy Member
Robert M. Kirby	0.06	
FTE Equivalent:	0.06	
Total Number:	1	

Names of Under Graduate students supported

<u>NAME</u>	<u>PERCENT SUPPORTED</u>
FTE Equivalent:	
Total Number:	

Student Metrics

This section only applies to graduating undergraduates supported by this agreement in this reporting period

The number of undergraduates funded by this agreement who graduated during this period: 0.00

The number of undergraduates funded by this agreement who graduated during this period with a degree in science, mathematics, engineering, or technology fields:..... 0.00

The number of undergraduates funded by your agreement who graduated during this period and will continue to pursue a graduate or Ph.D. degree in science, mathematics, engineering, or technology fields:..... 0.00

Number of graduating undergraduates who achieved a 3.5 GPA to 4.0 (4.0 max scale):..... 0.00

Number of graduating undergraduates funded by a DoD funded Center of Excellence grant for Education, Research and Engineering:..... 0.00

The number of undergraduates funded by your agreement who graduated during this period and intend to work for the Department of Defense 0.00

The number of undergraduates funded by your agreement who graduated during this period and will receive scholarships or fellowships for further studies in science, mathematics, engineering or technology fields: 0.00

Names of Personnel receiving masters degrees

<u>NAME</u>
Total Number:

Names of personnel receiving PhDs

<u>NAME</u>

Total Number:

Names of other research staff

<u>NAME</u>	<u>PERCENT SUPPORTED</u>
Eugene H. Payne	0.41
FTE Equivalent:	0.41
Total Number:	1

Sub Contractors (DD882)

1 a. Robert Haimes

1 b. MIT Department of Aeronautics and

77 Massachusetts Avenue 33-207

Cambridge MA 02139

Sub Contractor Numbers (c):

Patent Clause Number (d-1):

Patent Date (d-2):

Work Description (e):

Sub Contract Award Date (f-1): 8/1/12 12:00AM

Sub Contract Est Completion Date(f-2): 7/31/15 12:00AM

Inventions (DD882)

Scientific Progress

Approach

The thesis of this work that in turn drives our approach is that a visualization methodology for high-order finite element data that exploits the high-order nature of the data in its native form provides a visual representation that introduces no (or quantifiable) approximation error due to the visualization technique. There are many possible reasons why this thesis is beneficial to the high-order finite element community. The proposed visualization techniques (1) use the data in its native form, hence helping to allay the computational scientists' concern that information is being lost when current visualization techniques are applied; (2) allow the computational scientists to focus their efforts on elimination of other sources of error (modeling errors, numerical (simulation) errors, etc.) because it eliminates visualization approximation error; and (3) provide "ground truth" images to which low-order visualization techniques applied to high-order finite element data can be compared. To facilitate our efforts, we continue to build upon our current interactive, nVIDIA (www.nvidia.com) Graphical Processing Unit (GPU) based high-order FEM visualization software called EIVis ("Element Visualizer"). A concrete outcome of this work, in addition to peer-reviewed publications targeted to the applied mathematics, visualization and application-domain communities, is the implementation and dissemination of our results within this publicly available software package.

Scientific Opportunities and Barriers

Many current scientific visualization techniques applied to higher-order solutions are inadequate when used for knowledge extraction and assistance in reducing the error budget because they transform high-order data to low-order representations for visualization purposes – a process which in and of itself adds "visualization error" to the error budget. The scientist is currently burdened with determining whether or not an anomaly found in an image generated by a visualization technique is from the modeling and discretization assumptions made as part of the simulation or as part of the visualization technique. In most cases that burden is high. In this proposed work we will examine several common scientific visualization techniques used for depicting scalar field values from the results of high-order finite element (continuous Galerkin and discontinuous Galerkin) simulations. We propose to create "highorder cognizant" visualization algorithms and strategies for high-order continuous and discontinuous data on planar and curved elements.

Significance

The proposed research impacts three areas: the mathematical sciences, the computer sciences, and the interdisciplinary bridge lying between these two areas. The high-order finite element community will benefit from this effort through the proposed development of algorithms, which accurately and efficiently render simulation results. The visualization community will benefit through the exposure to the high-order finite element community and, in particular, the numerical methods prevalently found there. Other current projects funded by ARO's program in computational mathematics that use high order finite elements will find immediate utility of the algorithms and implementations discussed herein (e.g. projects funded at Brown University, RPI, etc.).

Accomplishments

This report summarizes three years of ARO funding on the current project. In the first year, a graduate student (Mr. Yichen Zhou) began working on the project. His focus during the first year was learning the EIVis software and beginning to examine mathematical mechanisms to accelerate ray-tracing of high-order discontinuous Galerkin fields, in particular whether high-order root finding combined with interval arithmetic can be used within our EIVis GPU environment. Mr. Zhou produced a project report for his efforts and gave a group presentation of his work. After assessment by both PIs of his work and performance on the project, Mr. Zhou was removed from the project.

To bridge our efforts while searching for a new student, we engaged two post doctoral researchers, one on the Utah side and one at MIT. On the MIT side, Dr. Marshall Galbraith began working on "EIVis hardening". The hardening effort consisted of streamlining the compilation procedure, fixing issues relayed to us by MIT users (MIT Project X users who provided feedback during the evaluation states of the work), working on in-code commenting and on documentation. He has also helped to strategize on how to make EIVis more cross platform stable. His efforts with respect to hardening of the software continued until the end of the project, and led to a second release of the software which was better documented and more stable.

On the Utah side, Dr. Shankar Sastry worked on more theoretical components of the EIVis work – namely questions arising from meshes. The original question posted to Dr. Sastry was a visualization question: could we accelerate the visualization of high-order methods by making assumptions on the inverse mapping used between world-space (engineering space) and the reference space (where basis functions are defined). In the course of investigating this topic, Dr. Sastry was able to prove some properties about quadratic element interpolation that did not exist previously in the literature. A paper on this work was published at the International Meshing Roundtable (IMR), October 2013. The title, author list and abstract are as follows:

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Abstract: Interpolation techniques are used to estimate function values and their derivatives at those points for which a numerical solution of any equation is not explicitly evaluated. In particular, the shape functions are used to interpolate a solution (within an element) of a partial differential equation obtained by the finite element method. Mesh generation and quality improvement are often driven by the objective of minimizing the bounds on the error of the interpolated solution. For linear elements, the error bounds at a point have been derived as a composite function of the shape function values at the point and its distance from the element's nodes. We extend the derivation to quadratic triangular elements and visualize the bounds for both the function interpolant and the interpolant of its derivative. The maximum error bound for a function interpolant within an element is computed using the active set method for constrained optimization. For the interpolant of the derivative, we visually observe that the evaluation of the bound at the corner vertices is sufficient to find the maximum bound within an element. We characterize the bounds and develop a mesh quality improvement algorithm that optimizes the bounds through the movement (r-refinement) of both the corner vertices and edge nodes of high-order meshes.

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In addition, on the Utah side, Mr. Gene Payne (software developer here at the SCI Institute) and Ms. Harshitha Venkata (graduate student) worked with Dr. Galbraith on the hardening and deployment of the code. Mr. Payne led the architecting work while Ms. Venkata tested the software on various use cases provided along with the software.

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Conclusions

Visualization is often employed as part of the simulation science pipeline. It is the lens through which scientists often examine their data for deriving new science, and is the lens used to view modeling and discretization interactions within their simulations. As such, visualization techniques need to be designed not only to elucidate the features or phenomena of interest within the data, but also to be compatible and complementary with the type and means of generating the data. One such category of simulation data, high-order finite element methods (also known as spectral/hp element methods) using either the continuous Galerkin or discontinuous Galerkin formulation, has reached a level of sophistication such that they are now commonly applied to a diverse set of real-life engineering problems. Visualizations of high-order finite element results that do not respect the a priori knowledge of how the data were produced and which do not provide a quantification of the visual error produced may undermine the scientific process as isolating where errors and assumptions are introduced into the process is critical.

Technology Transfer

**Visualization of Discontinuous Galerkin Based High-Order Methods
Final Report**

Proposal Number W911NF1210375

**Professor Robert M. Kirby, School of Computing, University of Utah and Mr. Robert Haimes,
Department of Aeronautics & Astronautics Massachusetts Institute of Technology**

Objective

The discontinuous Galerkin method (DGM) has become, in recent times, one of the most widely researched and utilized discretization methodologies for solving problems in science and engineering. Fundamentally based upon the mathematical framework of variational methods, the DG methodology provides hope that computationally fast, efficient and robust methods can be constructed for solving real-world problems. Through a combination of a dual path to convergence – allowing naturally both conforming and non-conforming (hanging node) non-overlapping discretizations of the solution domain combined with (possibly non-uniform) polynomial enrichment (also known as p-refinement) – the DG methodology provides a rich mathematical starting point for the development of domain-specific solvers. By not requiring that the solution be continuous across element boundaries, the DGM provides a flexibility that can be exploited both for geometric and solution adaptivity and for parallelization.

The goal of this proposal is to research, develop and implement visualization techniques that are specifically designed to respect the mathematical nature of high-order DGM simulation fields. As our intended use for this research is both simulation debugging and scientific exploratory visualization, we seek methods and implementations that act on high-order DG data directly, that are interactive, and that have quantifiable visual error. This effort is designed to build upon our previous research into the visualization of high-order (continuous Galerkin) finite element fields, but presents a distinct challenge in that we must now consider the additional computational (e.g. precision) and visualization challenges of element-wise discontinuities.

Approach

The thesis of this work that in turn drives our approach is that a visualization methodology for high-order finite element data that exploits the high-order nature of the data in its native form provides a visual representation that introduces no (or quantifiable) approximation error due to the visualization technique. There are many possible reasons why this thesis is beneficial to the high-order finite element community. The proposed visualization techniques (1) use the data in its native form, hence helping to allay the computational scientists' concern that information is being lost when current visualization techniques are applied; (2) allow the computational scientists to focus their efforts on elimination of other sources of error (modeling errors, numerical (simulation) errors,

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Personnel Funded

- Robert M. Kirby (Utah) – PI
- Robert Haimes (MIT) – co-PI
- Mr. Yichen Zhou (Utah) – graduate student (1st year of project)
- Ms. Harshitha Venkata – graduate student (3rd year of project)
- Dr. Marshall Galbraith (MIT) – post-doc
- Dr. Shankar Sastry (Utah) – post-doc
- Mr. Gene Payne – software developer

Conclusions

Visualization is often employed as part of the simulation science pipeline. It is the lens through which scientists often examine their data for deriving new science, and is the lens used to view modeling and discretization interactions within their simulations. As such, visualization techniques need to be designed not only to elucidate the features or phenomena of interest within the data, but also to be compatible and complementary with the type and means of generating the data. One such category of simulation data, high-order finite element methods (also known as spectral/*hp* element methods) using either the continuous Galerkin or discontinuous Galerkin formulation, has reached a level of sophistication such that they are now commonly applied to a diverse set of real-life engineering problems. Visualizations of high-order finite element results that do not respect the *a priori* knowledge of how the data were produced and which do not provide a quantification of the visual error produced may undermine the scientific process as isolating where errors and assumptions are introduced into the process is critical.

Technology Transfer

None to report at this time.